Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Round 1 - Algebraic Word Problems

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Farmer John started with eleven cows. He then bought fifty-seven more cows, but lost four of them. Then, he sold one-fourth of his cows to Farmer Sally. How many cows is Farmer John left with?

2. The sum of three consecutive integers is 17 more than the largest of the three integers. What is the product of the three integers?

3. Consider a collection of 72 coins, consisting of nickels, dimes and quarters. Their total is worth \$13.20, and there are three times as many quarters as nickels. How many more nickels do we need to add to the collection in order for the value of the nickels in the collection to be one-fourth of the total value of the collection?

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Round 2 - Number Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. What is the smallest square number divisible by 24?

2. Let a be the greatest common divisor of 238 and 308, and let b be the greatest common divisor of 552 and 576. Compute the least common multiple of a and b.

3. Find the sum of the distinct prime factors of 5015015005.

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Round 3 - Operations on Numerical Fractions, Decimals, Percents, and Percentage Word Problems

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Compute

$$1.3 + 0.\overline{6}\left(\frac{0.27}{0.45}\right)$$

Express your answer as a decimal.

2. A new car was on sale at a discount of 20%. Through a special deal, Bob was able to purchase the car for \$93.60 with an additional 10% discount on the already discounted price. How much did the car cost originally without either of the discounts?

3. School A and School B held a joint athletics event. Exactly $\frac{2}{11}$ of the students from School A played soccer, and exactly 76% of all of the students who played soccer were from School B. Out of all of the students, 15% were from School B and did not play soccer. What fraction of the total number of students were from School A and played soccer? Express your answer as a common fraction.

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Round 4 - Set Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. How many nonempty subsets of $A = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89\}$ contain only prime numbers?

2. Let A be the set of odd positive integers less than 30. Let B be the set of positive integers less than 30 divisible by 3. Let C be the set of positive integers less than 30 divisible by 5. Compute the sum of the elements in

$$(A \cap B) \cup (A \cap C)$$

3. A group of students were surveyed about whether they could speak English, Spanish, or French. All of the students could speak at least one of the languages. There were 29 students who could not speak English. There were 32 students who could not speak Spanish. There were 40 students who could not speak French. If the difference between the number of students who could speak exactly one language and the number of students who could speak all three languages was 37, how many students were surveyed?

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Evaluate $0.\overline{20} + 0.\overline{0023}$. Express your answer as a fraction.

2. How many positive integers less than 1000 are divisible by both 12 and 21?

3. It takes 5 hours for 9 high school students to solve 15 math questions. It takes 2 hours for 30 middle school students to solve 4 math questions. A group of high school and middle school students solve 54 questions in 9 hours. If there are 13 high school students in the group, how many middle school students are in the group?

4. Let $A = \{1, 3, 6, 7, 8, 9\}, B = \{2, 4, 6, 7, 9\}, C = \{3, 5, 8, 9\}$. How many elements are in

$$(A \cap C) \cup B$$

5. How many strings of length 5 composed of only 0s and 1s are there such there are no two 1s in a row?

6. Suppose

$$a + 4b = 12$$
$$4a + b = 33$$

Compute $a + b$.

7. Compute the area of a triangle with side lengths 18,41,41.

8. Evaluate

$$(1 - 2 + 3) + (2 - 3 + 4) + (3 - 4 + 5) + \cdots + (8 - 9 + 10)$$

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Team Round Answer Sheet

ANSWERS

1. _____

2. _____

3. _____

4. _____

5. _____

6. _____

7. _____

8. _____

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Answer Key

Round 1 - Evaluation and Order of Operations

1. 48
2. 720
3. 68

Round 2 - Number Theory

1. 144
2. 168
3. 36

Round 3 - Fractions...

1. 1.7
2. 130
3. $\frac{51}{520}$

Round 4 - Set Theory

1. 31
2. 105
3. 64

Team Round

1. $\frac{227}{1111}$
2. 11
3. 25
4. 7
5. 13
6. 9
7. 360
8. 44

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Round 1 - Algebraic Word Problems

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Farmer John started with eleven cows. He then bought fifty-seven more cows, but lost four of them. Then, he sold one-fourth of his cows to Farmer Sally. How many cows is Farmer John left with?

Solution: He starts with 11 cows, adds 57, and subtracts 4 to get 64 cows. Then, one-fourth of 64 is 16, so $64 - 16 = \boxed{48}$ cows.

2. The sum of three consecutive integers is 17 more than the largest of the three integers. What is the product of the three integers?

Solution: The three consecutive integers can be represented as $a-1, a, a+1$. Then,

$$(a-1) + a + (a+1) = (a+1) + 17$$

$$\Rightarrow 3a = a + 18$$

$$\Rightarrow 2a = 18$$

$$\Rightarrow a = 9$$

.

The three numbers are 8,9,10, so their product is $\boxed{720}$.

3. Consider a collection of 72 coins, consisting of nickels, dimes and quarters. Their total is worth \$13.20, and there are three times as many quarters as nickels. How many more nickels do we need to add to the collection in order for the value of the nickels in the collection to be one-fourth of the total value of the collection?

Solution: Let n, d, q represent the number of nickels, dimes, and quarters, respectively, in the original collection. Then, we have

$$n + d + q = 72$$
$$5n + 10d + 25q = 1320$$
$$q = 3n$$

Substituting $q = 3n$, we get

$$4n + d = 72$$
$$80n + 10d = 1320 \Rightarrow 8n + d = 132$$

Subtracting the equations, we get $4n = 60 \Rightarrow n = 15$. We want to see how many nickels x we need to add so that

$$4(5(n + x)) = 1320 + 5x$$
$$\Rightarrow 20(15 + x) = 1320 + 5x$$
$$\Rightarrow 15x = 1020$$
$$\Rightarrow x = \boxed{68}$$

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Round 2 - Number Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. What is the smallest square number divisible by 24?

Solution: We have 24 is divisible by 8, and every square number divisible by 8 is also divisible by 16. We also have 24 is divisible by 3, and every square number divisible by 3 is also divisible by 9. Thus, every square number divisible by 24 is divisible by $9 \cdot 16 = 144$. Thus, $\boxed{144}$ is the smallest square number divisible by 24.

2. Let a be the greatest common divisor of 238 and 308, and let b be the greatest common divisor of 552 and 576. Compute the least common multiple of a and b.

Solution: We have $\gcd(238, 308) = \gcd(238, 308 - 238) = \gcd(238, 70) = \gcd(238 - 3 \cdot 70, 70) = \gcd(28, 70) = 14$. Also, $\gcd(552, 576) = \gcd(576 - 552, 576) = \gcd(24, 576) = 24$. Then, we have $\text{lcm}(14, 24) = 2^3 \cdot 3 \cdot 7 = \boxed{168}$.

3. Find the sum of the distinct prime factors of 5015015005.

Solution: We can first factor $5015015005 = 5 \cdot 1003003001$. We notice 1, 3, 3, 1 are binomial coefficients, so we can rewrite it as $10^9 + 3 \cdot 10^6 + 3 \cdot 10^3 + 1 = (10^3 + 1)^3 = 1001^3$. We can factor $1001 = 7 \cdot 11 \cdot 13$, so $5015015005 = 5 \cdot 7^3 \cdot 11^3 \cdot 13^3$. The sum of the distinct prime factors is $5 + 7 + 11 + 13 = \boxed{36}$.

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Round 3 - Operations on Numerical Fractions, Decimals, Percents, and Percentage Word Problems

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Compute

$$1.3 + 0.\overline{6}\left(\frac{0.27}{0.45}\right)$$

Express your answer as a decimal.

Solution: This is equal to

$$\begin{aligned} & 1.3 + \frac{2}{3}\frac{27}{45} \\ &= 1.3 + \frac{2}{3}\frac{3}{5} \\ &= 1.3 + \frac{2}{5} \\ &= 1.3 + 0.4 \\ &= \boxed{1.7} \end{aligned}$$

2. A new car was on sale at a discount of 20%. Through a special deal, Bob was able to purchase the car for \$93.60 with an additional 10% discount on the already discounted price. How much did the car cost originally without either of the discounts?

Solution: If x is the originally cost of the car, the price after the two discounts is $(0.8)(0.9)x$. Then, setting $(0.8)(0.9)x = 93.60$, we get that $x = \boxed{130}$.

3. School A and School B held a joint athletics event. Exactly $\frac{2}{11}$ of the students from School A played soccer, and exactly 76% of all of the students who played soccer were from School B. Out of all of

the students, 15% were from School B and did not play soccer. What fraction of the total number of students were from School A and played soccer? Express your answer as a common fraction.

Solution: Let x be the fraction of the total number of student who were from School A and played soccer. Then, $\frac{9}{2}x$ is the fraction of total students who were from School A and did not play soccer. Also $\frac{76}{24}x = \frac{19}{6}x$ is the fractions of total students who were from School B and played soccer. Then, we have

$$\begin{aligned} x + \frac{9}{2}x + \frac{19}{6}x + 15\% &= 1 \\ \Rightarrow \frac{52}{6}x + \frac{3}{20} &= 1 \\ \Rightarrow x &= \boxed{\frac{51}{520}} \end{aligned}$$

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Round 4 - Set Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. How many nonempty subsets of $A = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89\}$ contain only prime numbers?

Solution: There are 5 primes numbers in A. Thus, there are $2^5 = 32$ subsets of A containing only prime numbers. We subtract 1 for the empty set to get $\boxed{31}$ nonempty subsets of A containing only primes numbers.

2. Let A be the set of odd positive integers less than 30. Let B be the set of positive integers less than 30 divisible by 3. Let C be the set of positive integers less than 30 divisible by 5. Compute the sum of the elements in

$$(A \cap B) \cup (A \cap C)$$

Solution: We have $A \cap B$ is the set of odd positive integers less than 30 divisible by 3. Also, $A \cap C$ is the set of odd positive integers less than 30 divisible by 5. Thus, their union is $\{3, 5, 9, 15, 21, 25, 27\}$. The sum of the elements in this set is $\boxed{105}$.

3. A group of students were surveyed about whether they could speak English, Spanish, or French. All of the students could speak at least one of the languages. There were 29 students who could not speak English. There were 32 students who could not speak Spanish. There were 40 students who could not speak French. If the difference between the number of students who could speak exactly one language and the number of students who could speak all three languages was 37, how many students were surveyed?

Solution: If we add the number of students who cannot speak English, the number of students who cannot speak Spanish, and the number of students who cannot speak French, we have that students who speak exactly one language are counted twice and students who speak exactly 2 languages are counted once. Thus, if we add the number of students who speak all three languages and subtract the number of students who speak exactly one language, each students is counted once, so we get the total. Thus, is equivalent to subtracting the difference between the number of studnets who could speak exactly one language and the number of students who could speak all three languages. Thus, the total number of students surveyed is $29 + 32 + 40 - 37 = \boxed{64}$.

Worcester County Mathematics League
Junior VarsityMeet 2 - December 13, 2023
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Evaluate $0.\overline{20} + 0.\overline{0023}$. Express your answer as a fraction.

Solution: This is equal to $\frac{20}{99} + \frac{23}{9999} = \frac{2020+23}{9999} = \frac{2043}{9999} = \boxed{\frac{227}{1111}}$.

2. How many positive integers less than 1000 are divisible by both 12 and 21?

Solution: The numbers that are divisible by both 12 and 21 are the numbers divisible by $\text{lcm}(12, 21) = 84$. The number of multiples of 84 less than 1000 is $\lfloor \frac{1000}{84} \rfloor = \boxed{11}$.

3. It takes 5 hours for 9 high school students to solve 15 math questions. It takes 2 hours for 30 middle school students to solve 4 math questions. A group of high school and middle school students solve 54 questions in 9 hours. If there are 13 high school students in the group, how many middle school students are in the group?

Solution: We have that in 1 hour, 1 high school student can solve $\frac{15}{5 \cdot 9} = \frac{1}{3}$ questions. Also, in 1 hour, 1 middle school student can solve $\frac{4}{2 \cdot 30} = \frac{1}{15}$ questions. Then, in 9 hours, the 13 high school students solve $\frac{9 \cdot 13}{3} = 39$ questions, so the middle school students solve $54 - 39 = 15$ questions. In order to solve 15 questions in 9 hours, there needs to be $\frac{15}{9 \cdot \frac{1}{15}} = \boxed{25}$ middle school students.

4. Let $A = \{1, 3, 6, 7, 8, 9\}, B = \{2, 4, 6, 7, 9\}, C = \{3, 5, 8, 9\}$. How many elements are in

$$(A \cap C) \cup B$$

Solution: We have $A \cap C = \{3, 8, 9\}$. Then, $(A \cap C) \cup B = \{2, 3, 4, 6, 7, 8, 9\}$, which is $\boxed{7}$ elements.

5. How many strings of length 5 composed of only 0s and 1s are there such there are no two 1s in a row?

Solution: There is 1 string with no 1s. There are 5 strings with one 1. There are 6 possible strings with two 1s. There is 1 possible string with three 1s. There are no possible strings with more than three 1s. Thus, there are $1 + 5 + 6 + 1 = \boxed{13}$ possible strings. In general, the number of strings with no consecutive 1s are the Fibonacci numbers.

6. Suppose

$$a + 4b = 12$$
$$4a + b = 33$$

Compute $a + b$.

Solution: Adding we get $5a + 4b = 45$. Dividing by 5, we get $a + b = \boxed{9}$.

7. Compute the area of a triangle with side lengths 18,41,41.

Solution: With the side with length 18 as the base, we compute the height. Drawing the altitude, we see a right triangle with hypotenuse 41 and one leg $18/2 = 9$. By Pythagoras, the height is than $\sqrt{41^2 - 9^2} = 40$. Thus, the area is $\frac{18 \cdot 40}{2} = \boxed{360}$.

8. Evaluate

$$(1 - 2 + 3) + (2 - 3 + 4) + (3 - 4 + 5) + \cdots + (8 - 9 + 10)$$

Solution: We see that this is equal to $2 + 3 + 4 + \cdots + 9 = \boxed{44}$.

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Answer Key

Round 1 - Evaluation and Order of Operations

1. 48
2. 720
3. 68

Round 2 - Number Theory

1. 144
2. 168
3. 36

Round 3 - Fractions...

1. 1.7
2. 130
3. $\frac{51}{520}$

Round 4 - Set Theory

1. 31
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Team Round

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